Modeling of one-dimensional weakly nonlinear waves that propagate in media with arbitrary dissipation and dispersion mechanisms

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In his paper [J. Acoust. Soc. Am. **77**, 2050 (1985)] Blackstock presented a generalized Burgers equation for the propagation of one-dimensional weakly nonlinear waves in various media. His results, and the approach he employed there, however, are limited to harmonic waves. In this paper, we present a general approach to model nonlinear waves of more general wave forms that propagate in media with arbitrary absorption and dispersion relations. The resulting equation is again called the generalized Burgers equation (to follow the terminology of the literature). It is found that steady shock solutions for various media can be described by the corresponding simplified version of the equation. An efficient numerical method by means of spectral analysis is developed for solving the generalized Burgers equation. Typical results exemplified by the case of a sinusoidal wave source are also reported in this paper.

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I. INTRODUCTION

Propagation of one-dimensional finite-amplitude waves in thermoviscous fluids is commonly described by the classical Burgers equation [1],

$$u_x - b u_{t't'} = (\beta/c_0^2) u u_{t'} \quad , \tag{1}$$

where u is particle velocity, x is distance from the source, $t' = t - x/c_0$ is retarded time (t is actual time), c_0 is small-signal sound speed, β is the coefficient of nonlinearity, and b is proportional to the coefficients of viscosity and heat conduction. The second order time derivative $u_{t't'}$ is a typical term for thermoviscous medium. The corresponding small-signal (i.e., when the wave amplitude is small) absorption coefficient α expressed in the frequency domain is given by $\alpha = b\omega^2$, where ω is the source frequency. The dispersion coefficient, however, is zero in this particular case.

When the medium cannot be characterized as thermoviscous, that is, when the dispersion is not negligible and α is not proportional to ω^2, it is still possible to derive a Burgers-like equation. Observing that many of the weakly nonlinear waves in various media have similar forms, Blackstock [2] generalized the classical Burgers equation to

$$u_x + L[u] = \frac{\beta}{c_0^2} u u_{t'} \quad , \tag{2}$$

where $L[u]$ is a linear operator that describes the small-signal absorption and dispersion properties of the medium. Following Blackstock, we will call Eq. (2) the generalized Burgers equation hereafter. The generalized Burgers equation can be applied to media with arbitrary dispersion and absorption relations, even though in some cases these two relations are known only empirically. When u is time harmonic, i.e., $u \sim \exp(i\omega t' - \zeta x)$, Blackstock showed that $L = \zeta u$, where ζ is a function related to the absorption-dispersion relations of the particular medium. His argument, however, is limited to time-harmonic signals. Our goal in this paper is to generalize his result to cover wave forms of more general shapes in media with arbitrary absorption-dispersion relations. The rest of this paper is organized as follows: In Sec. II, we present a general approach to derive the generalized Burgers equations in various media. The stationary solutions for some of these equations are presented in Sec. III. It is found that most of the equations derived do not possess any closed form solution. However, they can be integrated numerically. Section IV is devoted to a discussion of the numercial solutions of the generalized Burgers equation.

II. GENERALIZED BURGERS EQUATION

To generalize Blackstock's result, we first observe that Eq. (2) can be expressed in the frequency domain in the form

$$F_x(\omega, x) + \zeta(\omega) F(\omega, x) = i \frac{\beta}{c_0^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega', x) F(\omega - \omega', x) \omega' d\omega' \quad , \tag{3}$$

where F is the Fourier component of u:

$$u(t', x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega, x) e^{i\omega t'} d\omega \quad . \tag{4}$$

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Note that the second term on the left hand side of Eq. (3) has the same form as the corresponding one derived by Blackstock for harmonic waves. Although Blackstock has demonstrated this result only for harmonic waves, the same argument can be applied to the cases when we have wave forms of more general shapes. This can be established in the following. Neglecting the nonlinear term on the right hand side of Eq. (3) (i.e., when the signal is weak), we obtain a partial differential equation for $F(\omega, x)$. The general solution of that equation is $F(\omega, x) = g(\omega)\exp(-\zeta x)$, where $g(\omega)$ is an arbitrary function of ω. Substituting F back into Eq. (4), it is obvious that Eq. (3) is valid for waves which satisfy the condition $u(t', x) \sim \int g(\omega)\exp(i\omega t' - \zeta x)$; i.e., as long as the spatial dependence of u is exponentially decaying when the amplitude of u is weak.

Next taking the inverse Fourier transform of Eq. (3), we obtain

$$L[u] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt'' u(x, t'') K(t' - t'') \quad , \tag{5}$$

where the kernel K is the inverse Fourier transform of ζ, i.e.,

$$K(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \zeta(\omega) e^{i\omega t} \quad . \tag{6}$$

Formally, the function $\zeta(\omega)$ is composed of two functions, namely,

$$\zeta(\omega) = \alpha(\omega) - i\gamma(\omega) \quad , \tag{7}$$

where α is the small-signal absorption coefficient and γ is the dispersion relation

$$\gamma(\omega) = \omega\left(\frac{1}{c_0} - \frac{1}{c_{\rm ph}(\omega)}\right) \quad .$$

The symbol $c_{\rm ph}(\omega)$ in the above formula is the phase velocity at frequency ω. The absorption and dispersion relations are not independent; they are related to each other by the Kramers-Kronig relations. The Kramers-Kronig relations can be written in various equivalent forms. The following version presented by O'Connell, Jaynes, and Miller [3] has been found to be convenient in our discussion:

$$G_1(\omega) = \frac{2}{\pi}{\rm P}\int_0^{\infty} \frac{\omega' G_2(\omega')}{\omega'^2 - \omega^2} d\omega' \quad ,$$

$$G_2(\omega) = -\frac{2}{\pi}{\rm P}\int_0^{\infty} \frac{\omega G_1(\omega')}{\omega'^2 - \omega^2} d\omega' \quad ,$$

$$c_{\rm ph}(\omega) = [\rho_0 G_1(\omega)]^{-1/2} \quad ,$$

$$\alpha(\omega) = [\rho_0 c_{\rm ph}(\omega)/2]\, \omega G_2(\omega) \quad ,$$

where ρ_0 is the ambient density of the medium and P means the Cauchy principal value. Hence, after one determines the absorption coefficient α for a certain medium, either theoretically or empirically, one can then derive the corresponding dispersion relation γ (or vice versa) by means of the above Kramers-Kronig relations.

Many of the well-known nonlinear wave equations can be reproduced by means of Eqs. (2) and (5). For example, for a thermoviscous fluid where $\alpha = b\omega^2$, $\gamma = 0$, the kernel K is proportional to the second order derivative of the Dirac δ function. This can be proved as follows:

$$K(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b\omega^2 e^{i\omega t} d\omega$$

$$= -\frac{b}{\sqrt{2\pi}}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

$$= -b\sqrt{2\pi}\frac{\partial^2}{\partial t^2}\delta(t) \quad .$$

Substituting this kernel in Eq. (5) thus reduces $L[u]$ to $-bu_{t't'}$ and the classical Burgers equation, Eq. (1), is recovered. Similarly, if the medium exibits single relaxation such that $\alpha = m\omega^2\tau/2c_0(1 + \omega^2\tau^2)$, $\gamma = m\omega^3\tau^2/2c_0(1 + \omega^2\tau^2)$, where m is the strength of relaxation, and τ is the relaxation time, we can prove that (see Appendix A)

$$L[u] = -\frac{m}{2c_0}\frac{\partial}{\partial t'}\int_{-\infty}^{t'} \frac{\partial u}{\partial t''} e^{-(t'-t'')/\tau} dt'' \quad . \tag{8}$$

Substituting Eq. (8) in Eq. (2), we reproduce the well-known nonlinear wave equation in a single relaxing medium [4].

For complex media, distributions of relaxation processes are possible. This mechanism is commonly charaterized by a distribution function of relaxation times $g(\tau/\tau_0)$. (Some authors use distribution functions of relaxation frequencies instead.) The parameter τ_0 denotes an arbitrarily chosen reference relaxation time. Some of these distribution functions are discrete, while others are continuous, depending on the particular medium. For instance, the major relaxation processes in air are the vibrational and rotational relaxation of N_2, O_2, and CO_2. The distribution function is obviously discrete. However, for media constituted by complex molecules, such as biological tissues, polymers, etc., the distribution functions are continuous. The distribution function g can be derived either by physical reasoning or by phenomenological arguments. The author [5] has discussed the special case when $g = \tau_0/\tau$, a function which has been proved to be appropriate to describe ultrasonic absorption of biomedical soft tissues [6,7]. Using the above outlined Fourier transform approach, we can derive the following generalized Burgers equation for tissues (see Appendix B for derivation):

$$\frac{\partial u}{\partial x} - b\frac{\partial^2 u}{\partial t'^2} - \frac{m_0}{2c_0}\frac{\partial}{\partial t'}\int_{-\infty}^{t'} \frac{\partial u}{\partial t''} K(t' - t'') dt'' = \frac{\beta}{2\rho_0 c_0^2}\frac{\partial u^2}{\partial t'} \quad , \tag{9}$$

where τ_l and τ_s are the largest and smallest relaxation times in the medium. The kernel K in this particular case is found to be

$$K(t) = E_1(t/\tau_l) - E_1(t/\tau_s) \quad , \tag{10}$$

where E_1 is the exponential integral of the first order [8].

As a final example, let us consider the dissipation due to a boundary layer. It is well known that in a boundary layer $\alpha = a\sqrt{\omega}$ and $\gamma = -a\sqrt{\omega}$ [9,10], where a is a constant related to the coefficient of viscosity, thermal conductivity, and geometry factors for the boundaries. Applying the inverse Fourier transform on the corresponding ζ, we thus recover (see Appendix C)

$$L[u] = a\sqrt{\frac{2}{\pi}} \int_{-\infty}^{t'} u_{t''}(x,t'') \frac{1}{\sqrt{t'-t''}} dt'' \quad . \tag{11}$$

III. STATIONARY SOLUTIONS (STEADY SHOCK SOLUTIONS)

By stationary solution (or steady shock solution), we mean the particular solution which does not change its wave shape with propagation distance. The stationary solution for the generalized Burgers equation can be derived by dropping the first term on the left hand side of Eq. (2):

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'' u(x,t'') K(t'-t'') = \frac{\beta}{c_0^2} u u_{t'} \quad . \tag{12}$$

Equation (12) is a nonlinear integrodifferential equation. In general, the solution of Eq. (12) requires information about the kernel K. Analytical solutions for cases with only thermoviscous dissipation [Eq. (1)] and with only a single relaxation mechanism [Eq. (8)] have been found (see, for examples, [4]). For some other kinds of kernels which do not possess analytical solutions, we can still derive a fairly general numerical algorithm to solve the corresponding integral equations. Note that in the two generalized Burger equations derived for the single relaxation and multiple relaxation cases, there is a common factor $\partial/\partial t'$ in front of the integrals. This allows us to simplify the equations by integrating with respect to t' once. Let us, for example, consider the multiple relaxation case. Integrating the equation with respect to t', we obtain

$$v^2 = 1 - 2D \int_{-\infty}^{y} K(y-y') v_{y'}(y') dy' \quad . \tag{13}$$

In Eq. (13) the particle velocity u and the retarded time t' have been replaced, respectively, by dimensionless variables $v = u/u_0$ and $y = \omega_0 t'$. Here u_0 and ω_0 are, respectively, the characteristic particle velocity and frequency at the source. The constant $D = m_0 c_0 / 2\beta u_0$ is the ratio of the relaxation strength to the nonlinear parameters. When $D \ll 1$, the relaxation effect is negligible, compared to the nonlinear effect (and vice versa for $D \gg 1$). Note also that the integration constant has been chosen to satisfy the boundary conditions $v \to \pm 1$ as $y \to \pm\infty$. No closed form solution can be found for Eq. (13). [See [5] for a discussion on the asymptotic solution of Eq. (13) by means of the strained coordinate perturbation method.] We can, however, solve the equation numerically by means of the succesive iteration method [11]. First, integrating by parts the integral on the right hand side of Eq. (13) yields

$$\frac{1-v^2}{2D_r} = v - \int_{-\infty}^{y} \mathcal{K}(y-y') v dy' \quad , \tag{14}$$

where $\mathcal{K} = (dK/dy')/\ln(\tau_l/\tau_s)$ and $D_r = D\ln(\tau_l/\tau_s)$. Note that Eq. (14) is translational invariant, i.e., the equation is unchanged under the transformation $y \mapsto y + y_a$, where y_a is an arbitrary constant. Without loss of generality, we can therefore choose the particular solution that passes through the origin. Next, we apply the successive iteration method to Eq. (14). This yields the following algorithm:

$$v^{(n+1)} = \int_{-\infty}^{y} \mathcal{K}(y-y') v^{(n)} dy' - \frac{1 - {v^{(n)}}^2}{2D_r} \quad , \tag{15}$$

where the integer index n denotes the value of v at the nth iteration step. To start the iteration, we use $v^{(0)} = \tanh(y/D_r)$. This corresponds to the asymptotic solution ($D_r \gg 1$) in the single relaxation case. Analysis of the exact solution of the corresponding single relaxation equation,

$$y = \ln\left[\frac{(1+v)^{D-1}}{(1-v)^{D+1}}\right] \quad ,$$

shows that the wave form becomes multivalued when $D < 1$. Thus, we limit the iteration scheme to $D_r > 1$. Experience shows that the scheme works very well in most cases.

Figure 1 shows the wave form of a strong shock ($D = 1.5$) for a single relaxing medium. Wave-form steepening occurs because of the nonlinear propagation effect. It also shows asymmetry around the origin because of the dispersion effect. In contrast, wave forms of the steady shock for a thermoviscous medium are symmetric when the dispersion γ is zero. For comparison purposes, we have included also the exact solution (the dashed curve)

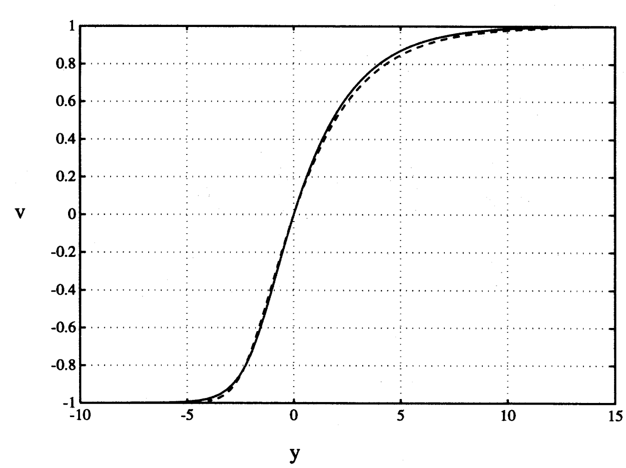

FIG. 1. Steady shock solution for the generalized Burgers equation in medium with single relaxation process at $D = 1.5$. The solid curve is the numerical solution and the dashed curve is the exact solution.

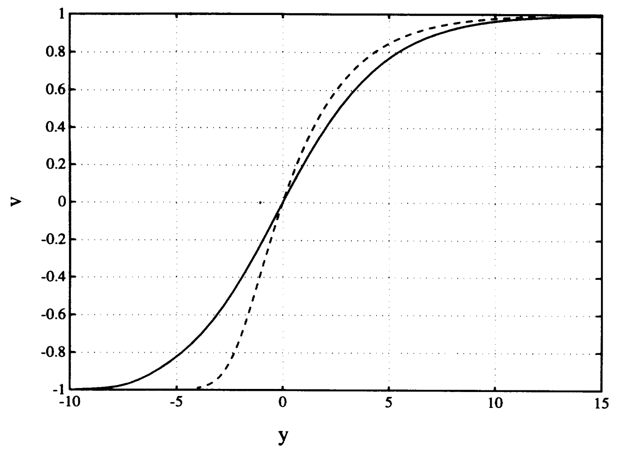

FIG. 2. Steady shock solution for the generalized Burgers equation in a medium with multiple relaxation processes at $D = 1.5, r = 0.1$ The solid curve is the numerical solution and the dashed curve is the exact solution when $D = 1.5, r = 1$ (single relaxation).

in the figure. It is found that the numerical solution is quite close to the exact one; this provides the credibility of the iteration scheme. Figure 2 shows the numerical solution of Eq. (15) for a multiple relaxing medium. The wave has the same strength as the one in Fig. 1, but with a broad bandwidth of the relaxation distribution ($\tau_s/\tau_l = 0.1$). We have included also the steady shock solution for a single relaxing medium with $\tau_s/\tau_l = 1$ (the dashed curve) for comparison. It is found that the wave form for the multiple relaxing case shows a wider half-width compared to the single relaxation one. Furthermore, the degree of asymmetry becomes less significant. This implies that distributing the relaxation processes over a wider frequency band strengthens the opposition to steepening. Indeed, these figures show some interesting interplay between the nonlinear effect and the relaxation effect.

IV. NUMERICAL SOLUTIONS

In this section, we present the numerical solutions to the full form of the generalized Burgers equation (2). For demonstration purposes, we concentrate on media carrying multiple relaxation mechanisms. Application of the method to other media should be obvious. In principle, both time-domain and frequency-domain (or spectral) approaches can be used. Both approaches have their own merits. For the time-domain approach, one solves the nonlinear integrodifferential equation directly by using the finite-difference method to transform the original equation into a system of algebric equations. The advantage of this approach is that it applies both to periodic and nonperiodic wave forms. The disadvantage of this method, however, is very time consuming, because of the fact that the resulting algebric system generally involves a large nonsparse matrix in each iteration step which may not be solved easily.

For periodic wave forms, however, the frequency-domain approach is more appropriate. In the frequency-domain approach, one uses the discrete Fourier transform to convert Eq. (2) to a system of coupled ordinary differential equations for the harmonic components. The system of equations can then be solved numerically, for example, by the Runge-Kutta method. The wave form at various spatial distances from the source is then determined by summing the contributions from all of the frequency components. Our experience shows that this method can be very fast and accurate provided that the amplitude of the wave form is not too strong. When the wave is very strong, however, a very large number of harmonic components must be retained if the shocks are to be described faithfully; consequently, the computation time is long. The finite-difference method may be more suitable in this situation. The numerical scheme solving Burgers-like equations in the frequency domain was first developed by Trivett and Van Buren [12]. However, their algorithm is limited to a signal without any dc component. We have modified their alogrithms so that they can be applied to wave forms of general shapes.

We now apply the spectral method to the generalized Burgers equation. First we expand v in a Fourier series:

$$v(\sigma, y) = \sum_{n=-\infty}^{\infty} v_n(\sigma) e^{iny} \quad , \tag{16}$$

$$v_n(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(\sigma, y) e^{-iny} dy \quad , \tag{17}$$

where $v_n(\sigma)$ is the nth harmonic frequency component of the wave form at spatial position σ. Again, we have replaced the physical distance x by a dimensionless distance $\sigma = x/x_s$, where $x_s = c_0^2/\beta u_0 \omega_0$ is the shock formation distance [1]. By using these expressions, we transform Eq. (2) into a system of differential equations [which is basically the discretized version of Eq. (3)],

$$\frac{dv_n}{d\sigma} + \zeta_n x_s v_n = \frac{in}{2} v_n \circ v_n \quad . \tag{18}$$

The function ζ_n is the absorption-dispersion function evaluated at the frequency of the nth harmonic component. The term on the right hand side of Eq. (18) represents the convolution

$$v_n \circ v_n = \sum_{m=-\infty}^{\infty} v_m v_{m-n}^* \quad , \tag{19}$$

where the superscript $*$ denotes complex conjugate. For practical numerical purposes, we limit the number of harmonics included in the calculation. We let n have values ranging from $-N$ to N, where N is a large enough positive integer, e.g., $N = 125$ or 256. Since v is a physical measurable quantity, it is a real function. This implies $v_{-n} = v_n^*$ and thus Eq. (19) becomes

$$v_n \circ v_n = 2v_0 v_n + \sum_{m=1}^{n-1} v_m v_{n-m} + 2 \sum_{m=n+1}^{N} v_m v_{m-n}^* \quad . \tag{20}$$

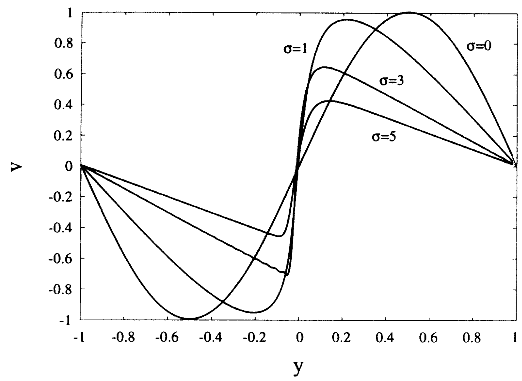

FIG. 3. Wave-form distortion for an originally sinusoidal wave at various distances from the source: $\sigma = 0$, $\sigma = 1$, $\sigma = 3$, $\sigma = 5$. $D = 0.5$, $r = 0.1$.

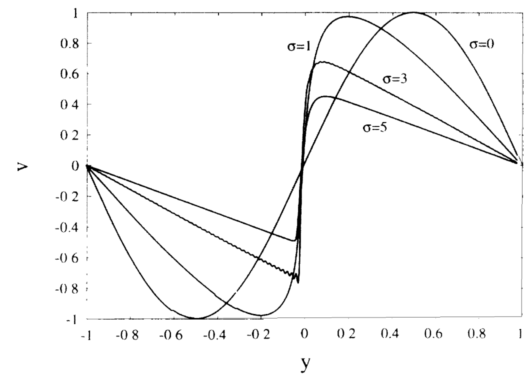

FIG. 5. Wave-form distortion for an originally sinusoidal wave at various distances from the source: $\sigma = 0$, $\sigma = 1$, $\sigma = 3$, $\sigma = 5$. $D = 0.25$, $r = 0.01$.

We can integrate Eqs. (18) and (20) step by step, beginning from the initial wave form $v(0, y) = f(y)$. After solving v_n, wave forms at various σ are then determined by the formula

$$v(\sigma, y) = v_0(0) + \sum_{n=1}^{N} 2|v_n(\sigma)| \cos(ny + \phi_n) \quad . \tag{21}$$

The amplitude $|v_n|$ and the phase ϕ_n can be determined by $v_n = |v_n| e^{i\phi_n}$.

Note that the first term v_0 in Eq. (21) represents the dc term in the wave form. Geometrically, it is the total area enclosed by the wave form. Its value is a constant, according to the area preservation law proved in [5]. For the sinusoidal wave source, it is zero. For a pulsed wave form, e.g., a temporal Gaussian wave, it is nonzero.

Figures 3–5 show the wave forms obtained by the spectral method when the medium has multiple relaxation mechanism. (Numerical solution of the correspoding generalized Burgers equation for the boundary layer problem has been studied by Sugimoto [14] from a different approach.) Each set of four curves shows the progressive distortion and decay of the wave as σ increases from 0 to 5. The first two sets are for a weak wave (Fig. 3) and a stronger wave (Fig. 4), respectively. Comparison shows that, as expected, the stronger wave has a shorter rise time and the wave form is more asymmetric. The ringing on the $\sigma = 3$ curve in Fig. 4 is Gibbs's phenomenon, which illustrates the difficulty encountered for the spectral method when we are dealing with very strong waves. Notice that at $\sigma = 5$ the ringing has practically disappeared. At this distance, relaxation has gained the upper hand in its battle with nonlinear steepening, and the shock has dispersed to the point that the number of harmonic components retained (125 in this case) is enough to give a true picture of the wave form. The wave forms shown in Fig. 5 have the same strength as the ones in Fig. 4, but the bandwidth of the relaxation processes is an order of magnitude greater. The nearly complete absence of ringing for the profiles in Fig. 5 implies that wider bandwidth does strengthen the absorption capability of the medium.

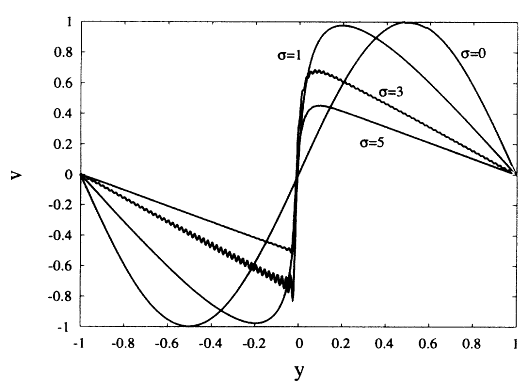

FIG. 4. Wave-form distortion for an originally sinusoidal wave at various distances from the source: $\sigma = 0$, $\sigma = 1$, $\sigma = 3$, $\sigma = 5$. $D = 0.25$, $r = 0.1$.

V. CONCLUSIONS

We have presented in this paper a general approach to derive weakly nonlinear wave equations in media with arbitrary absorption and dispersion relations. Numerical algorithms for solving both the steady shock and progressive waves have been reported. We believe that the approaches presented here could be useful for studies of nonlinear waves propagating in media with novel absorption-dispersion relations.

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APPENDIX A

Substituting $\alpha = m\omega^2\tau/2c_0(1+\omega^2\tau^2)$ and $\gamma = m\omega^3\tau^2/2c_0(1+\omega^2\tau^2)$ in Eq. (7) yields

$$\zeta = \frac{m}{2c_0}\frac{\omega^2\tau}{1+i\omega\tau} \quad . \tag{A1}$$

Defining $y = \omega\tau$ and $T = t/\tau$, we can write the inverse Fourier transform of this ζ function as

$$-\frac{m}{2c_0}\frac{1}{\sqrt{2\pi}\tau}\frac{\partial^2}{\partial T^2}\int_{-\infty}^{\infty} e^{iyT}\frac{1}{1+iy}dy \quad . \tag{A2}$$

The integrand in Eq. (A2) has a simple pole at $z = i$ in the z complex plane. When $T \geq 0$, we can enclose this pole by using a semihemisphere contour in the upper half-plane. The residual of the contour integral equals $2\pi\exp(-T)$. When $T < 0$, it is easy to show that the integral vanishes by using a lower half-plane contour. Therefore,

$$K(t) = -\frac{m}{2c_0\tau}\sqrt{2\pi}\frac{\partial^2}{\partial T^2}\left\{e^{-T}\theta(T)\right\} \tag{A3}$$

$$= -\frac{m}{2c_0\tau}\sqrt{2\pi}e^{-T}\left\{\theta(T) - 2\delta(T) + \delta'(T)\right\} \ , \tag{A4}$$

where θ is the Heaviside step function and δ is the Dirac delta function. Substituting this kernel into Eq. (5) followed by some algebra thus yields Eq. (8).

APPENDIX B

Let the distribution function of the relaxation time be $g \sim 1/\tau$. The absorption coefficient for the media with continuous distribution of relaxation processes should therefore be an integral of the absorption coefficient due to single relaxation, multiplied by the distribution function, i.e.,

$$\alpha = M\int_{\tau_s}^{\tau_l}\frac{\omega^2\tau}{1+\omega^2\tau^2}\frac{1}{\tau}d\tau$$

$$= M\omega\left(\arctan\omega\tau_l - \arctan\omega\tau_s\right) \quad , \tag{B1}$$

where M is a constant containing m_0, c_0, τ_0, etc. Similarly, the dispersion relation for the multiple relaxation case should be

$$\gamma = M\int_{\tau_s}^{\tau_l}\frac{\omega^3\tau^2}{1+\omega^2\tau^2}\frac{1}{\tau}d\tau \tag{B2}$$

$$= M\omega\frac{1}{2}\ln\left(\frac{1+\omega^2\tau_l^2}{1+\omega^2\tau_s^2}\right) \quad . \tag{B3}$$

The kernel in this case is composed of two terms $K(t) = K_1(t) - iK_2(t)$. The first term K_1 is the inverse Fourier transform of Eq. (B1), i.e.,

$$\sqrt{2\pi}K_1(t) = \int_{-\infty}^{\infty} M\omega\left[\arctan(\omega\tau_l) - \arctan(\omega\tau_s)\right]e^{i\omega t}d\omega$$

$$= -M\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\frac{\arctan\omega\tau_l - \arctan\omega\tau_s}{\omega}e^{i\omega t}d\omega \quad .$$

Since $\arctan(\omega\tau/\omega)$ is an odd function, we obtain

$$\sqrt{2\pi}K_1(t) = -2M\frac{\partial^2}{\partial t^2}\int_0^{\infty}\frac{\arctan\omega\tau_l - \arctan\omega\tau_s}{\omega}\cos\omega t d\omega \ . \tag{B4}$$

Similarly, the intergal for the dispersion relation is

$$\sqrt{2\pi}K_2(t) = -iM\frac{\partial^2}{\partial t^2}\int_0^{\infty}\frac{\ln(1+\omega^2\tau_l^2) - \ln(1+\omega^2\tau_s)}{\omega}\sin\omega t d\omega \quad . \tag{B5}$$

Knowing that (see [13])

$$\int_0^{\infty}\frac{\arctan x}{x}\cos\omega x dx = \frac{\pi}{2}E_1(\omega)\theta(\omega) \quad ,$$

$$\int_0^{\infty}\frac{\ln(1+x^2)}{x}\sin\omega x dx = \pi E_1(\omega)\mathrm{sgn}(\omega) \quad ,$$

where sgn is the sign function and E_1 is the exponential integral of the first order, we thus obtain

$$K(t) = -\sqrt{2\pi}M\frac{\partial^2}{\partial t^2}\left\{\left[E_1(t/\tau_l) - E_1(t/\tau_s)\right]\theta(t)\right\} \ . \tag{B6}$$

Substituting Eq. (B6) in Eq. (5) and carrying out the integration thus yields Eq. (9).

APPENDIX C

The function ζ for the boundary layer problem is $\zeta = a\sqrt{2i\omega}$ and the kernel is

$$K(t) = a\sqrt{\frac{i}{\pi}}\int_{-\infty}^{\infty}\sqrt{\omega}e^{i\omega t}d\omega$$

$$= a\sqrt{\frac{1}{i\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\sqrt{\omega}}d\omega \quad .$$

First let us consider the case when $t > 0$. Changing the variable $\omega t = u^2$, we can transform the above integral into a contour integral,

$$\frac{2}{\sqrt{t}}\int_c e^{iu^2}du \quad , \tag{C1}$$

where the contour is composed of two straight lines: L_1 coincides with the y axis on the complex plane, directed from $+\infty$ to the origin; L_2 is the positive x axis, directed from the origin to $+\infty$. The contributions from these two paths are

$$\int_{L_1} e^{iu^2}du = i\int_0^\infty \cos y^2 dy + \int_0^\infty \sin y^2 dy \quad ,$$
$$\int_{L_2} e^{iu^2}du = \int_0^\infty \cos y^2 dy + i\int_0^\infty \sin y^2 dy \quad .$$

Knowing that the Fresnel integrals on the right hand sides in the above formulas have the values $\int_0^\infty \cos y^2 dy = \int_0^\infty \sin y^2 dy = \sqrt{\pi/8}$, we therefore obtain

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\sqrt{\omega}}d\omega = 2\sqrt{\frac{i\pi}{t}} \quad . \tag{C2}$$

Next, when $t < 0$, the contour becomes as follows: L_1' coincides with the positive x axis, ranging from $+\infty$ to 0; L_2' coincides with the negative y axis, ranging from 0 to $-\infty$. Hence,

$$\begin{aligned}\int_c \frac{e^{i\omega t}}{\sqrt{\omega}}d\omega &= \frac{2}{\sqrt{-t}}(1+i)\int_0^\infty \cos y^2 dy \\ &\quad -\frac{2}{\sqrt{-t}}(1+i)\int_0^\infty \sin y^2 dy \\ &= 0 \quad .\end{aligned}$$

Therefore, combining the above results, we obtain

$$K(t) = 2a\frac{\partial}{\partial t}\left[\frac{1}{|t|^{1/2}}\theta(t)\right] \quad . \tag{C3}$$

Substituting Eq. (C3) into Eq. (5), we thus recover Eq. (11).

[1] See, for examples, M. J. Lighthill, in *Survey in Mechanics*, edited by G. K. Batchelor and R. M. Davies (Cambridge University Press, Cambridge, England, 1956), pp. 250–351; D. T. Blackstock, J. Acoust. Soc. Am. **36**, 534 (1964).

[2] D. T. Blackstock, J. Acoust. Soc. Am. **77**, 2050 (1985).

[3] M. O'Connell, E. T. Jaynes, and J. G. Miller, J. Acoust. Soc. Am. **69**, 696 (1981).

[4] S. I. Soluyan and R. V. Khokhlov, Sov. Phys. Acoust. **8**, 170 (1962).

[5] P. W. Li, Ph.D. dissertation, The University of Texas at Austin, 1993.

[6] E. L. Carstensen and H. P. Schwan, J. Acoust. Soc. Am. **31**, 305 (1959).

[7] H. Pauly and H. P. Schwan, J. Acoust. Soc. Am. **50**, 692 (1970).

[8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series Vol. **55** (U.S. GPO, Washington, D.C., 1964), Chap. 5.

[9] W. Chester, J. Fluid Mech. **18**, 44 (1964).

[10] D. T. Blackstock, in *American Institute of Physics Handbook*, 3rd ed., edited by D. E. Gray (McGraw-Hill, New York, 1972), pp. 3-183–3-205.

[11] R. H. Moore, in *Nonlinear Integral Equations*, edited by P. M. Anselone (University of Wisconsin Press, Madison, WI, 1964), pp. 65–98.

[12] D. H. Trivett and A. L. Van Buren, J. Acoust. Soc. Am. **69**, 943 (1981).

[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, corrected and enlarged ed. (Academic Press, New York, 1980), pp. 600, 614.

[14] N. Sugimoto, J. Fluid Mech. **225**, 631 (1964).